Mat 1. 2-timers prøve 17.5.2014.

Opgave 1.

$f(x) = 2\cos x - \sin 2x \; , \quad x \in \mathbb{R}.$

1. $f'(x) = -2\sin x - 2\cos 2x$.

   $f''(x) = -2\cos x + 4\sin 2x$.

   $f'''(x) = 2\sin x + 8\cos 2x$.

2. Med udviklingspunkt $x_0 = 0$ er

   $\underline{P_2(x)} = f(0) + f'(0)x + \frac{1}{2} f''(0) x^2 = \underline{2 - 2x - x^2}$.

3. Ifølge Taylors formel med $x_0 = 0$ findes et ξ mellem $\frac{1}{10}$ og 0, så

$f(\tfrac{1}{10}) = P_2(\tfrac{1}{10}) + \frac{1}{3!} f'''(\xi)\left(\frac{1}{10}\right)^3 = P_2(\tfrac{1}{10}) + \frac{1}{6}(2\sin\xi + 8\cos 2\xi)\frac{1}{10^3}$.

Benyttes $P_2(\frac{1}{10}) = \frac{179}{100} = 1{,}79$ i stedet for $f(\frac{1}{10})$ så er

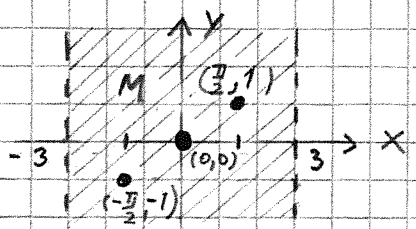
fejlen $\underline{|f(\tfrac{1}{10}) - P_2(\tfrac{1}{10})|} = |R_2(\tfrac{1}{10})| = \frac{1}{6\cdot 10^3}|2\sin\xi + 8\cos 2\xi| \leqq \frac{10}{6\cdot 10^3} = \underline{\frac{1}{600}}$,

idet $0 < 2\sin\xi + 8\cos 2\xi \leqq 2 + 8 = 10$, da $\xi \in [0; \frac{1}{10}]$.

Da $f(\frac{1}{10}) - P_2(\frac{1}{10}) = R_2(\frac{1}{10}) > 0$ og da $\frac{1}{600} = 0{,}001666\ldots < 0{,}0017$

fås $P_2(\frac{1}{10}) < f(\frac{1}{10}) < P_2(\frac{1}{10}) + 0{,}0017$. Altså $1{,}7900 < f(\frac{1}{10}) < 1{,}7917$.

Opgave 2.



$f(x,y) = \frac{1}{2}y^2 - y\sin x$.

$M = \{(x,y) \in \mathbb{R}^2 \mid -3 < x < 3\}$.

1. $f'_x(x,y) = -y\cos x \; , \quad f'_y(x,y) = y - \sin x$.

   $f''_{xx}(x,y) = y\sin x \; , \quad f''_{xy}(x,y) = f''_{yx}(x,y) = -\cos x \; , \quad f''_{yy}(x,y) = 1$.

2. $\nabla f(x,y) = (f'_x(x,y), f'_y(x,y)) = (-y\cos x, y - \sin x) = (0,0)$ og $(x,y) \in M$

$\Leftrightarrow (y = 0 \wedge \sin x = 0 \wedge x \in]-3;3[) \vee (\cos x = 0 \wedge y - \sin x = 0 \wedge x \in]-3;3[)$.

Hvilket giver punkterne $\underline{(0,0)}$, $\underline{(-\frac{\pi}{2}, -1)}$ og $\underline{(\frac{\pi}{2}, 1)}$.

<u>Disse tre punkter er således de eneste stationære punkter i M.</u>

3. Hesse-matricen for f i punktet (x,y) er

Opgave 2 fortsat.

$$\underline{H}(x,y)=\begin{bmatrix} f''_{xx}(x,y) & f''_{xy}(x,y) \\ f''_{xy}(x,y) & f''_{yy}(x,y)\end{bmatrix}=\begin{bmatrix} y\sin x & -\cos x \\ -\cos x & 1\end{bmatrix}.$$

Hvis f har lokalt minimum eller lokalt maksimum i et punkt af M, så må punktet være et stationært punkt i M, da f ikke har undtagelsespunkter i M.

$H(0,0)=\begin{bmatrix} 0 & -1 \\ -1 & 1\end{bmatrix}$. $\begin{vmatrix} -\lambda & -1 \\ -1 & 1-\lambda\end{vmatrix}=\lambda^2-\lambda-1=0 \Leftrightarrow \lambda=\begin{cases}\frac{1+\sqrt5}{2} & >0 \\ \frac{1-\sqrt5}{2} & <0\end{cases}$.

f har hverken lokalt minimum eller lokalt maksimum i det stationære punkt $(0,0)$.

$\underline{H}(-\frac{\pi}{2},-1)=\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}=\underline{H}(\frac{\pi}{2},1)$. $\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda\end{vmatrix}=(1-\lambda)^2=0 \Leftrightarrow \lambda=\begin{cases}1 & >0 \\ 1 & >0\end{cases}$.

f har egentligt lokalt minimum i de stationære punkter $(-\frac{\pi}{2},-1)$ og $(\frac{\pi}{2},1)$ med værdierne $f(-\frac{\pi}{2},-1)=-\frac12=f(\frac{\pi}{2},1)$.

Opgave 3.

Rumkurve K_r givet ved

$\underline{r}(t)=(e^t-e^{-t},\ e^t+e^{-t},\ 1-2t)$, $t\in[0;1]$.

1. $\underline{r}'(t)=(e^t+e^{-t},\ e^t-e^{-t},\ -2)$.

$|\underline{r}'(t)|=\sqrt{(e^t+e^{-t})^2+(e^t-e^{-t})^2+4}=\sqrt{2e^{2t}+2e^{-2t}+4}$

$=\sqrt2\sqrt{(e^t+e^{-t})^2}=\underline{\sqrt2(e^t+e^{-t})}$.

$\underline{L(K_r)}=\int_{K_r}ds=\int_0^1|\underline{r}'(t)|dt=\sqrt2\int_0^1(e^t+e^{-t})dt=\underline{\sqrt2(e-e^{-1})}$.

$\underline{V}(x,y,z)=(y,x,-2)$, $(x,y,z)\in\mathbb{R}^3$. (Første ordens)

2. $\underline{V}(\underline{r}(t))=(e^t+e^{-t},\ e^t-e^{-t},\ -2)$.

$\underline{V}(\underline{r}(t))\cdot\underline{r}'(t)=(e^t+e^{-t})^2+(e^t-e^{-t})^2+4=2e^{2t}+2e^{-2t}+4$.

$\underline{\mathrm{Tan}(\underline{V},K_r)}=\int_{K_r}\underline{V}\cdot\underline{t}\,ds=\int_0^1\underline{V}(\underline{r}(t))\cdot\underline{r}'(t)\,dt=\int_0^1(2e^{2t}+2e^{-2t}+4)\,dt$

$=\left[e^{2t}-e^{-2t}+4t\right]_0^1=\underline{e^2-e^{-2}+4}$.

3. Mat 1. 2-timers prøve 17. 5. 2014.

Opgave 3.

3. Da $\underline{V}(\underline{r}(t)) = (e^t + e^{-t}, e^t - e^{-t}, -2) = \underline{r}'(t)$ for alle $t \in [0;1]$ og da $\underline{r}(0) = (0,2,1)$, så er $K_{\underline{r}}$ den flowkurve for $\underline{V}$, der starter i punktet $(0,2,1)$ for $t=0$. (Jævnfør eksistens- og entydighedssætningen)

Opgave 4.

Massivt område Ω i rummet givet ved

$\underline{r}(u,v,w) = (u\cos w, u\sin w, v(2-u))$, hvor

$u \in [1;2]$, $v \in [0;1]$ og $w \in [0;\pi]$.

1. $f(x,y,z) = 1$ og $g(x,y,z) = \frac{y}{2}$.

$\underline{r}'_u(u,v,w) = (\cos w, \sin w, -v)$.

$\underline{r}'_v(u,v,w) = (\ 0\ ,\ 0\ ,\ 2-u)$.

$\underline{r}'_w(u,v,w) = (-u\sin w, u\cos w, 0)$.

$$J_{\underline{r}}(u,v,w) = \left|\underline{r}'_u\ \underline{r}'_v\ \underline{r}'_w\right| = \begin{vmatrix} \cos w & 0 & -u\sin w \\ \sin w & 0 & u\cos w \\ -v & 2-u & 0 \end{vmatrix}$$

$$= -(2-u)u < 0, \text{ da } u \in [1;2].$$

$$\text{Jacobi}_{\underline{r}}(u,v,w) = |J_{\underline{r}}(u,v,w)| = u(2-u) = 2u - u^2.$$

$$\underline{\int_\Omega f\,d\mu} = \int_\Omega d\mu = \int_{w=0}^{\pi}\left(\int_{v=0}^{1}\left(\int_{u=1}^{2}\text{Jacobi}_{\underline{r}}(u,v,w)\,du\right)dv\right)dw$$

$$= \int_{w=0}^{\pi}\left(\int_{v=0}^{1}\left(\int_{u=1}^{2}(2u-u^2)\,du\right)dv\right)dw = \pi\left[u^2 - \tfrac{1}{3}u^3\right]_1^2 = \underline{\frac{2\pi}{3}} = Vol(\Omega).$$

$$\underline{\int_\Omega g\,d\mu} = \int_\Omega \tfrac{1}{2}y\,d\mu = \int_{w=0}^{\pi}\left(\int_{v=0}^{1}\left(\int_{u=1}^{2}\tfrac{1}{2}u\sin w\cdot\text{Jacobi}_{\underline{r}}(u,v,w)\,du\right)dv\right)dw$$

$$= \int_{w=0}^{\pi}\left(\int_{v=0}^{1}\left(\int_{u=1}^{2}\sin w\,(u^2 - \tfrac{1}{2}u^3)\,du\right)dv\right)dw = \left[-\cos w\right]_{w=0}^{\pi}\cdot\left[\tfrac{1}{3}u^3 - \tfrac{1}{8}u^4\right]_{u=1}^{2}$$

$$= \underline{\frac{11}{12}}.$$

$\underline{V}(x,y,z) = (\frac{1}{2}z^2, \frac{1}{4}y^2, -2y)$, $(x,y,z) \in \mathbb{R}^3$.

4. Mat 1. 2-timers prøve 17.5.2014.

Opgave 4.

2. $\operatorname{div} \underline{V} = \underline{\nabla} \cdot \underline{V} = \frac{1}{2} y = g(x,y,z)$.

$\partial\Omega$ er den lukkede overflade af Ω orienteret med udadrettet enhedsnormalvektor.

Af Gauss' sætning fås da

$$\underline{\text{Flux}(\underline{V}, \partial\Omega)} = \int_\Omega \operatorname{div} \underline{V} \, d\mu = \int_\Omega g \, d\mu \underset{1.}{=} \underline{\frac{11}{12}}.$$

3. $\underline{U}(x,y,z)$, så $\operatorname{div} \underline{U} = k$ og $\text{Flux}(\underline{U}, \partial\Omega) = \int_{\partial\Omega} \underline{U} \cdot \underline{n}_{\partial\Omega} \, d\mu = 2\pi$.

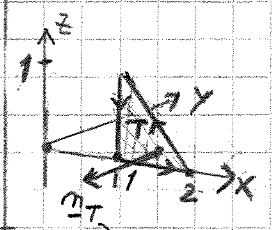
Af Gauss' sætning fås

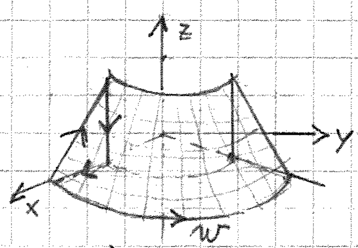
$$\text{Flux}(\underline{U}, \partial\Omega) = \int_\Omega \operatorname{div} \underline{U} \, d\mu = k \int_\Omega d\mu \underset{1.}{=} \frac{2k\pi}{3} = 2\pi \iff$$

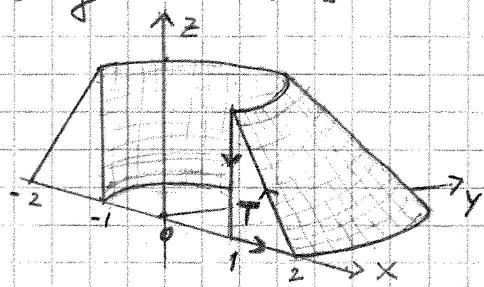
$k = \operatorname{div} \underline{U} = 3$. $\underline{\text{F.eks. } \underline{U}(x,y,z) = (x,y,z)}$.

4. Sættes $w = 0$ fås trekant-området T givet ved

$\underline{r}(u,v) = (u, 0, v(2-u))$, $u \in [1;2]$ og $v \in [0;1]$.







$\underline{r}'_u(u,v) = (1, 0, -v)$.

$\underline{r}'_v(u,v) = (0, 0, 2-u)$.

$\underline{N}(u,v) = \underline{r}'_u \times \underline{r}'_v = (0, u-2, 0) = (2-u)(0,-1,0)$, $u \in [1;2]$.

$\underline{\text{rot}} \, \underline{V} = \underline{\nabla} \times \underline{V} = (-2, z, 0)$.

$\underline{\text{rot}} \, \underline{V}(\underline{r}(u,v)) = (-2, v(2-u), 0)$

Med den valgte orientering af randkurven ∂T for T er $\underline{n}_T = (0,-1,0)$ (højrekonvention) og

$\underline{n}_T = \frac{\underline{N}(u,v)}{|\underline{N}(u,v)|}$ for $u \in [1;2[$. Af Stokes's sætning fås da

$$\underline{\text{Cirk}(\underline{V}, \partial T)} = \text{Flux}(\underline{\text{rot}}\,\underline{V}, T) = \int_T \underline{n}_T \cdot \underline{\text{rot}}\,\underline{V} \, d\mu$$

$$= \int_{v=0}^{1} \left(\int_{u=1}^{2} \underline{N}(u,v) \cdot \underline{\text{rot}}\,\underline{V}(\underline{r}(u,v)) \, du \right) dv = \int_{v=0}^{1} -v \left(\int_{u=1}^{2} (4 + u^2 - 4u) \, du \right) dv = \underline{-\frac{1}{6}}.$$