1. Mat 1. 2-timers prøve 10.5.2012

Opgave 1.

$f(x,y) = e^{x-y}$.

1. $f'_x(x,y) = e^{x-y}$, $f'_y(x,y) = -e^{x-y}$.
$f''_{xx}(x,y) = e^{x-y}$, $f''_{xy}(x,y) = -e^{x-y}$, $f''_{yy}(x,y) = e^{x-y}$.
$f'_x(0,0) = 1$, $f'_y(0,0) = -1$, $f''_{xx}(0,0) = 1$, $f''_{xy}(0,0) = -1$, $f''_{yy}(0,0) = 1$.

2. Med udviklingspunktet $(x_0, y_0) = (0,0)$ fås:

$$\underline{P_2(x,y)} = f(0,0) + f'_x(0,0)x + f'_y(0,0)y + \tfrac{1}{2}f''_{xx}(0,0)x^2 + f''_{xy}(0,0)xy + \tfrac{1}{2}f''_{yy}(0,0)y^2$$
$$= \underline{1 + x - y + \tfrac{1}{2}x^2 - xy + \tfrac{1}{2}y^2}.$$

3. Med udviklingspunkt $(x_0, y_0) = (a,a)$, $a \in \mathbb{R}$, på linien $y = x$ fås:

$$\underline{P_2(x,y)} = f(a,a) + f'_x(a,a)(x-a) + f'_y(a,a)(y-a)$$
$$+ \tfrac{1}{2}f''_{xx}(a,a)(x-a)^2 + f''_{xy}(a,a)(x-a)(y-a) + \tfrac{1}{2}f''_{yy}(a,a)(y-a)^2$$
$$= 1 + (x-a) - (y-a) + \tfrac{1}{2}(x-a)^2 - (x-a)(y-a) + \tfrac{1}{2}(y-a)^2$$
$$= 1 + x - y + \tfrac{1}{2}x^2 + \tfrac{1}{2}a^2 - ax - xy + ax + ay - a^2 + \tfrac{1}{2}y^2 + \tfrac{1}{2}a^2 - ay$$
$$= \underline{1 + x - y + \tfrac{1}{2}x^2 - xy + \tfrac{1}{2}y^2 \quad \text{for alle } a \in \mathbb{R}}.$$

Heraf ses, at de alle er ens.

Opgave 2.

$\underline{\underline{A}} = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}$, $a \in \mathbb{R}$. $\lambda_1 = a+1$, $E_{a+1} = \operatorname{span}\{(1,1)\}$; $\lambda_2 = a-1$, $E_{a-1} = \operatorname{span}\{(-1,1)\}$. $\underline{\underline{A}} = \underline{\underline{A}}^T \Rightarrow E_{a+1} \perp E_{a-1}$.

1. $q_1 = \frac{1}{\sqrt{2}}(1,1)$ er en ortonormal basis for E_{a+1} og
$q_2 = \frac{1}{\sqrt{2}}(-1,1) = \hat{q}_1$ er en ortonormal basis for E_{a-1}.

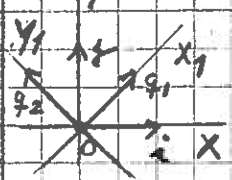
Egenvektorerne (q_1, q_2) er da en ortonormal basis for $(\mathbb{R}^2, \cdot)$.

Sættes $\underline{\underline{Q}} = [q_1\, q_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ og $\underline{\underline{\Lambda}} = \begin{bmatrix} a+1 & 0 \\ 0 & a-1 \end{bmatrix}$, så er

$\underline{\underline{Q}}$ egentlig ortogonal og $\underline{\underline{Q}}^{-1}\underline{\underline{A}}\,\underline{\underline{Q}} = \underline{\underline{Q}}^T\underline{\underline{A}}\,\underline{\underline{Q}} = \underline{\underline{\Lambda}}$.

Opgave 2 fortsat.

2. og 3. Sædvanligt retvinklet koordinatsystem $(O; \underline{i}, \underline{j})$ i planen. Et keglesnit er givet ved:



$$ax^2+ay^2+2xy=1 \Leftrightarrow [x\ y]\underbrace{\begin{bmatrix}a & 1\\ 1 & a\end{bmatrix}}_{\underline{\underline{A}}}\begin{bmatrix}x\\ y\end{bmatrix}=1,\quad a\in\mathbb{R}.$$

$(O; \underline{q}_1, \underline{q}_2)$, hvor $\underline{q}_1=\frac{1}{\sqrt{2}}(1,1)$ og $\underline{q}_2=\hat{\underline{q}}_1=\frac{1}{\sqrt{2}}(-1,1)$ er et nyt sædvanligt retvinklet koordinatsystem fremkommet ved en drejning af det givne koordinatsystem 45° om O.

$$\begin{bmatrix}x\\ y\end{bmatrix}=\underbrace{\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}}_{\underline{\underline{Q}}}\begin{bmatrix}x_1\\ y_1\end{bmatrix}.$$

Keglesnittets ligning i det nye koordinatsystem:

$$[x_1\ y_1]\,\underline{\underline{Q}}^T\underline{\underline{A}}\,\underline{\underline{Q}}\begin{bmatrix}x_1\\ y_1\end{bmatrix}=1 \underset{1.}{\Leftrightarrow} (a+1)x_1^2+(a-1)y_1^2=1.$$

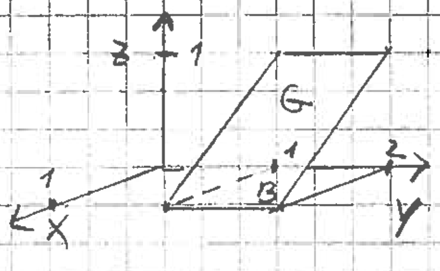
Dvs. en <u>hyperbel for $-1<a<1$</u> og
en <u>ellipse for $a>1$</u>.

Opgave 3.

$h(x,y)=1-x$, hvor $(x,y)\in B=\{(x,y)\mid 0\leq x\leq 1 \text{ og } 1\leq y\leq 2\}$.

Grafen G for h har ligningen:

$z=1-x \Leftrightarrow x+z=1$ for $(x,y)\in B$. Dvs G er et rektangel.



1. Parameterfremstilling for G

$\underline{r}(u,v)=(u,v,1-u)$, hvor

$u\in[0;1]$ og $v\in[1;2]$.

$\underline{r}'_u(u,v)=(1,0,-1)$

$\underline{r}'_v(u,v)=(0,1,0)$

$\underline{N}(u,v)=\underline{r}'_u\times\underline{r}'_v=(1,0,1)$. $\underline{\text{Jacobi}_{\underline{r}}(u,v)}=\|\underline{N}(u,v)\|=\underline{\sqrt{2}}$.

2. $f(x,y,z)=\frac{x+z}{y}$ er defineret for $y\neq 0$ og $(x,z)\in\mathbb{R}^2$.

Dvs $Dm(f)=\{(x,y,z)\in\mathbb{R}^3\mid y\neq 0\}$, hvilket er de to halvrum på begge sider af xz-planen.

Opgave 3 fortsat.

Da G ikke indeholder punkter af formen $(x,0,z)$, så tilhører G $Dm(f)$. (G ligger helt i det "højre" halvrum. Se figuren.)

3. $$\int_G f(x,y,z)\,d\mu = \int_{v=1}^{2}\left(\int_{u=0}^{1} f(\underline{r}(u,v))\,\text{Jacobi}_{\underline{r}}(u,v)\,du\right)dv$$

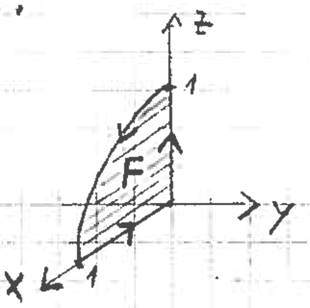
$$= \int_{v=1}^{2}\left(\int_{u=0}^{1} \frac{\sqrt{2}}{v}\,du\right)dv = \underline{\sqrt{2}\ln 2}\,.$$

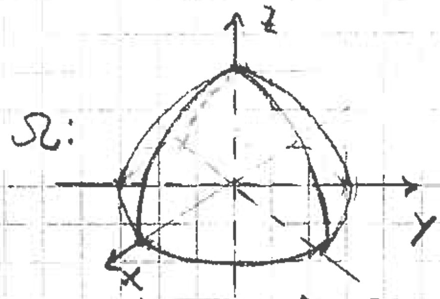
Opgave 4.

$\underline{V}(x,y,z) = (z^2, 5y, -2x)\,,\quad (x,y,z) \in \mathbb{R}^3\,.$

1. $\text{div}\,\underline{V} = \underline{\nabla}\cdot\underline{V} = \underline{5}\,,\quad \underline{\text{rot}}\,\underline{V} = \underline{\nabla}\times\underline{V} = \underline{(0, 2z+2, 0)}\,.$

2.





$F:\ \underline{r}(u,v) = (v(1-u^2), 0, u)\,,\quad u\in[0;1]\,,\quad v\in[0;1]\,.$

$\quad\underline{r}'_u(u,v) = (-2uv, 0, 1)$

$\quad\underline{r}'_v(u,v) = (1-u^2, 0, 0)$

$\underline{N}(u,v) = \underline{r}'_u \times \underline{r}'_v = (0, 1-u^2, 0)\,.$

Med den valgte orientering af randkurven ∂F for F er $\underline{n}_F = (0,1,0) = \frac{\underline{N}(u,v)}{\|\underline{N}(u,v)\|}$ (Højrekonvention).

Af Stokes' sætning fås

$\underline{\text{Cirk}(\underline{V}, \partial F)} = \text{Flux}(\underline{\text{rot}}\,\underline{V}, F) = \int_F \underline{n}_F \cdot \underline{\text{rot}}\,\underline{V}\,d\mu$

$$= \int_{v=0}\int_{u=0} \underline{N}(u,v)\cdot\underline{\text{rot}}\,\underline{V}(\underline{r}(u,v))\,du\,dv = \int_{v=0}^{1}\left(\int_{u=0}^{1}(1-u^2)(2u+2)\,du\right)dv$$

$$= \int_{u=0}^{1}(2u+2-2u^3-2u^2)\,du = \underline{\frac{11}{6}}\,.$$

Opgave 4 fortsat.

3. Parameterfremstilling for Ω.

$\underline{r}(u,v,w) = (v(1-u^2)\cos w, v(1-u^2)\sin w, u)$,

$u \in [0;1]$, $v \in [0;1]$, $w \in [0;2\pi]$.

$\underline{r}'_u(u,v,w) = (-2uv\cos w, -2uv\sin w, 1)$

$\underline{r}'_v(u,v,w) = ((1-u^2)\cos w, (1-u^2)\sin w, 0)$

$\underline{r}'_w(u,v,w) = (-v(1-u^2)\sin w, v(1-u^2)\cos w, 0)$

$\underline{\mathrm{Jacobi}_{\underline{r}}(u,v,w) =}$ $|\det([\underline{\underline{r}}'_u\ \underline{\underline{r}}'_v\ \underline{\underline{r}}'_w])| = \underline{v(1-u^2)^2}$.

4.

$$\underline{\mathrm{Vol}(\Omega)} = \int_\Omega 1\, d\mu = \int_{w=0}^{2\pi}\left(\int_{v=0}^{1}\left(\int_{u=0}^{1}\mathrm{Jacobi}_{\underline{r}}(u,v,w)\, du\right)dv\right)dw$$

$$= \int_{w=0}^{2\pi}\left(\int_{v=0}^{1}\left(\int_{u=0}^{1} v(1-u^2)^2\, du\right)dv\right)dw = 2\pi\int_{v=0}^{1} v\left(\int_{u=0}^{1}(1+u^4-2u^2)\,du\right)dv$$

$$= 2\pi\int_{v=0}^{1}\frac{8v}{15}\,dv = \underline{\frac{8}{15}\pi}.$$

5. $\partial\Omega$ er den lukkede overflade af Ω orienteret med udadrettet enhedsnormalvektor.

Af Gauss' sætning fås da

$$\underline{\mathrm{Flux}(V, \partial\Omega)} = \int_\Omega \mathrm{div}\,\underline{V}\, d\mu = 5\int_\Omega d\mu = 5\,\mathrm{Vol}(\Omega) = \underline{\frac{8}{3}\pi}.$$