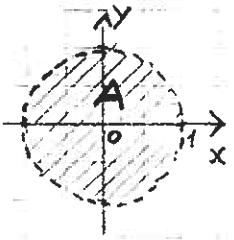
Mat 1 . 2-timers prøve 17.5.2010.

Opgave 1.

$f(x,y)$. Definitionsmængde $A = \{(x,y) \mid x^2+y^2<1\}$.



Af Maple-arket aflæses:

$f(0,0)=1$, $f'_x(x,y) = \frac{-2x}{1-x^2-y^2}$, $f'_y(x,y) = \frac{-2y}{1-x^2-y^2}$.

$f''_{xx}(x,y) = \frac{-2}{1-x^2-y^2} - \frac{4x^2}{(1-x^2-y^2)^2}$.

$f''_{yy}(x,y) = \frac{-2}{1-x^2-y^2} - \frac{4y^2}{(1-x^2-y^2)^2}$.

$f''_{xy}(x,y) = \frac{-4xy}{(1-x^2-y^2)^2}$.

1. $\underline{\nabla} f(x,y) = (f'_x(x,y), f'_y(x,y)) = (0,0) \Leftrightarrow (x,y) = (0,0) \in A$.
$\underline{(0,0) \text{ er således eneste stationære punkt for } f.}$

2. Da A er åben og da f ikke har undtagelsespunkter i A, kan et eventuelt ekstremum kun forekomme i det stationære punkt $(0,0)$. Hessian-matricen i $(0,0)$:

$$\underline{\underline{A}}(0,0) = \begin{bmatrix} f''_{xx}(0,0) & f''_{xy}(0,0) \\ f''_{yx}(0,0) & f''_{yy}(0,0) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}. \quad \lambda = -2 \text{ (dobbelt)}.$$

<u>Da $\underline{\underline{A}}(0,0)$ har to negative egenværdier, så har f egentligt maksimum i det stationære punkt $(0,0)$.</u>

3. $P_2(x,y) = f(0,0) + f'_x(0,0)x + f'_y(0,0)y + \frac{1}{2}\left(f''_{xx}(0,0)x^2 + 2f''_{xy}(0,0)xy + f''_{yy}(0,0)y^2\right)$
$= \underline{1 - x^2 - y^2}$.

Opgave 2.

1. Af LA lemma 8.41 fås

$$x^2+y^2+z^2+2xz = [x\ y\ z]\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}. \text{ Dvs. } \underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \underline{\underline{A}}^T.$$

2. $$P_{\underline{\underline{A}}}(\lambda) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)((1-\lambda)^2-1) = -\lambda(1-\lambda)(2-\lambda) = 0 \Leftrightarrow \lambda = \begin{cases} 2 \\ 1 \\ 0 \end{cases}.$$

Da $\underline{\underline{A}}$ er symmetrisk, er de tre egenvektorrum E_2, E_1 og E_0 parvis ortogonale.

$$\underline{\underline{A}} - 2\underline{\underline{I}} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad E_2 = \text{span}\{(1,0,1)\}.$$

Opgave 2 fortsat.

$q_1 = \frac{1}{\sqrt{2}}(1,0,1)$ er en ortonormal basis for E_2.

$\underline{\underline{A}} - \underline{\underline{I}} = \begin{bmatrix} 0&0&1 \\ 0&0&0 \\ 1&0&0 \end{bmatrix} \sim \begin{bmatrix} 1&0&0 \\ 0&0&1 \\ 0&0&0 \end{bmatrix}$. $E_1 = \text{span}\{(0,1,0)\}$.

$q_2 = (0,1,0)$ er en ortonormal basis for E_1.

$q_3 = q_1 \times q_2 = \frac{1}{\sqrt{2}}(-1,0,1)$ er en ortonormal basis for E_0.

Egenvektorerne (q_1, q_2, q_3) er da en ortonormal basis for $\mathbb{R}^3$ udstyret med det sædvanlige skalarprodukt.

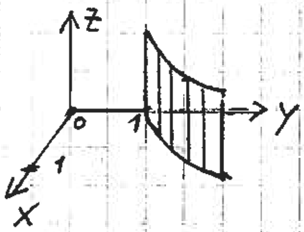
$\underline{\underline{Q}} = [\,\underline{q}_1\ \underline{q}_2\ \underline{q}_3\,] = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$ er egentlig ortogonal (LA eks. 8.13)

og da $\underline{\underline{Q}}^T \underline{\underline{A}}\, \underline{\underline{Q}} = \underline{\underline{Q}}^{-1} \underline{\underline{A}}\, \underline{\underline{Q}} = \begin{bmatrix} 2&0&0 \\ 0&1&0 \\ 0&0&0 \end{bmatrix}$ (LA sætning 8.33), så er

$[x_1, y_1, z_1]\, \underline{\underline{Q}}^T \underline{\underline{A}}\, \underline{\underline{Q}} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = [x_1, y_1, z_1] \begin{bmatrix} 2&0&0 \\ 0&1&0 \\ 0&0&0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = 2x_1^2 + y_1^2$ for alle $x_1, y_1, z_1 \in \mathbb{R}$.

Opgave 3.

$\mathcal{F}$: $\mathcal{L}$ er $y = \cosh x$, $x \in [0;1]$ og $z \in [0;1]$.



1. Parameterfremstilling for $\mathcal{F}$:

$\overrightarrow{OP} = \underline{r}(u,v) = (u, \cosh u, v)$, $u \in [0;1]$, $v \in [0;1]$.

$\underline{r}'_u(u,v) = (1, \sinh u, 0)$.

$\underline{r}'_v(u,v) = (0, 0, 1)$.

$\underline{N}(u,v) = \underline{r}'_u(u,v) \times \underline{r}'_v(u,v) = (\sinh u, -1, 0)$.

$\underline{\text{Jacobi}_{\underline{r}}(u,v)} = \|\underline{N}(u,v)\| = \sqrt{1 + \sinh^2 u} = \underline{\cosh u}$.

2. $\underline{\text{Ar}(\mathcal{F})} = \int_{\mathcal{F}} dS = \int_{v=0}^{1} \left(\int_{u=0}^{1} \|\underline{N}(u,v)\|\, du \right) dv = \int_{v=0}^{1} \left(\int_{u=0}^{1} \cosh u\, du \right) dv$

$= \underline{\sinh(1)}$.

3. Rumkurve $\mathcal{K}$ givet ved parameterfremstillingen

$\overrightarrow{OP} = \underline{r}(u) = (u, \cosh u, \frac{1}{2})$, $u \in [0;1]$.

Parameterkurven på $\mathcal{F}$ svarende til $v = \frac{1}{2}$.

Opgave 3 fortsat.

$\underline{r}'(u) = (1, \sinh u, 0)$.

$\underline{\text{Jacobi}_{\underline{r}}(u)} = \|\underline{r}'(u)\| = \sqrt{1+\sinh^2 u} = \underline{\cosh u}$.

$\underline{\int_{\mathcal{K}} 2z\, ds} = \int_{u=0}^{1} \|\underline{r}'(u)\|\, du = \int_{u=0}^{1} \cosh u\, du = \underline{\sinh(1)}$.

Opgave 4.

Massivt område Ω i rummet givet ved
$\vec{OP} = \underline{r}(u,v,w) = (u\cos v, u\sin v, w(1-u^3))$, hvor
$u \in [0;1]$, $v \in [0;\pi]$ og $w \in [0;1]$.

1. $J_{\underline{r}}(u,v,w) = \left| \underline{\underline{r}}'_u \ \underline{\underline{r}}'_v \ \underline{\underline{r}}'_w \right| = \begin{vmatrix} \cos v & -u\sin v & 0 \\ \sin v & u\cos v & 0 \\ -3wu^2 & 0 & 1-u^3 \end{vmatrix} = u(1-u^2)$.

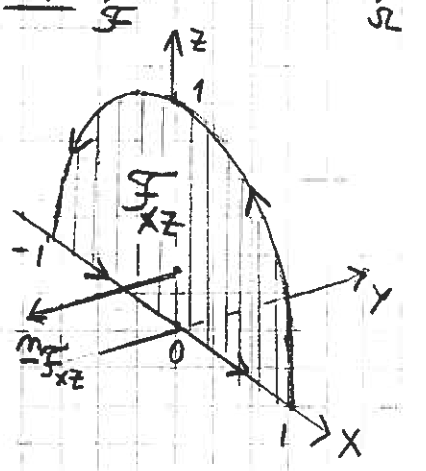
$\text{Jacobi}_{\underline{r}}(u,v,w) = |J_{\underline{r}}(u,v,w)| = u(1-u^3)$, da $u \in [0;1]$.

$\underline{\text{Vol}(\Omega)} = \int_{\Omega} d\Omega = \int_{w=0}^{1}\left(\int_{v=0}^{\pi}\left(\int_{u=0}^{1} u(1-u^3)\,du\right)dv\right)dw = \pi\int_{u=0}^{1}(u-u^4)\,du$

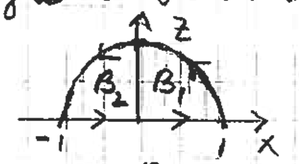
$= \pi\left[\frac{u^2}{2} - \frac{u^5}{5}\right]_{u=0}^{1} = \pi\left(\frac{1}{2}-\frac{1}{5}\right) = \underline{\frac{3\pi}{10}}$.

Vektorfelt $\underline{V}(x,y,z)$, $\operatorname{div}\underline{V} = \underline{\nabla}\cdot\underline{V} = 5$, $\operatorname{rot}\underline{V} = \underline{\nabla}\times\underline{V} = (0,-2,0)$.
$\mathcal{F} = \partial\Omega$ orienteret med udadrettet enhedsnormalvektor.

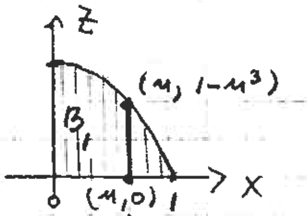
2. Af Gauss' sætning fås:
$\underline{\Phi} = \int_{\mathcal{F}} \underline{V}\cdot\underline{n}\, dS = \int_{\Omega} \operatorname{div}\underline{V}\, d\Omega = 5\int_{\Omega} d\Omega = 5\,\text{Vol}(\Omega) = \underline{\frac{3\pi}{2}}$.



Med den valgte orientering af rand=kurven $\mathcal{K} = \delta\mathcal{F}_{XZ}$ er $\underline{n}_{\mathcal{F}_{XZ}} = (0,-1,0)$.
(Højrekonvention.)



$\mathcal{F}_{XZ} = B_1 \cup B_2$



Parameterfremstilling for B_1: $\underline{r}(u,0,w) = (u,0,w(1-u^3)) = (u,0,0) + w(0,0,1-u^3)$,
hvor $u \in [0;1]$ og $w \in [0;1]$. Dvs. $B_1 = \{(x,z) \mid 0 \leq x \leq 1 \wedge 0 \leq z \leq 1-x^3\}$.

Opgave 4 fortsat.

3. Af Stokes' sætning fås:

Cirkulationen $= \oint_{\mathcal{K}} \underline{V} \cdot \underline{t}\, ds = \int_{\mathcal{F}_{xz}} \underline{n}_{\mathcal{F}_{xz}} \cdot \underline{\mathrm{rot}}\, \underline{V}\, dS = \int_{\mathcal{F}_{xz}} 2\, dS$

$= 2\, Ar(\mathcal{F}_{xz}) \underset{Ar(B_1) = Ar(B_2)}{=} 4\, Ar(B_1) = 4 \int_{x=0}^{1} \left(\int_{z=0}^{1-x^3} dz \right) dx = 4 \int_{x=0}^{1} (1 - x^3)\, dx$

$= 4 \left[x - \frac{x^4}{4} \right]_{x=0}^{1} = 4\left(1 - \frac{1}{4}\right) = \underline{3}.$